Rubbing Shoulders With Newton:
A New Look at a Fundamental Constant of Nature

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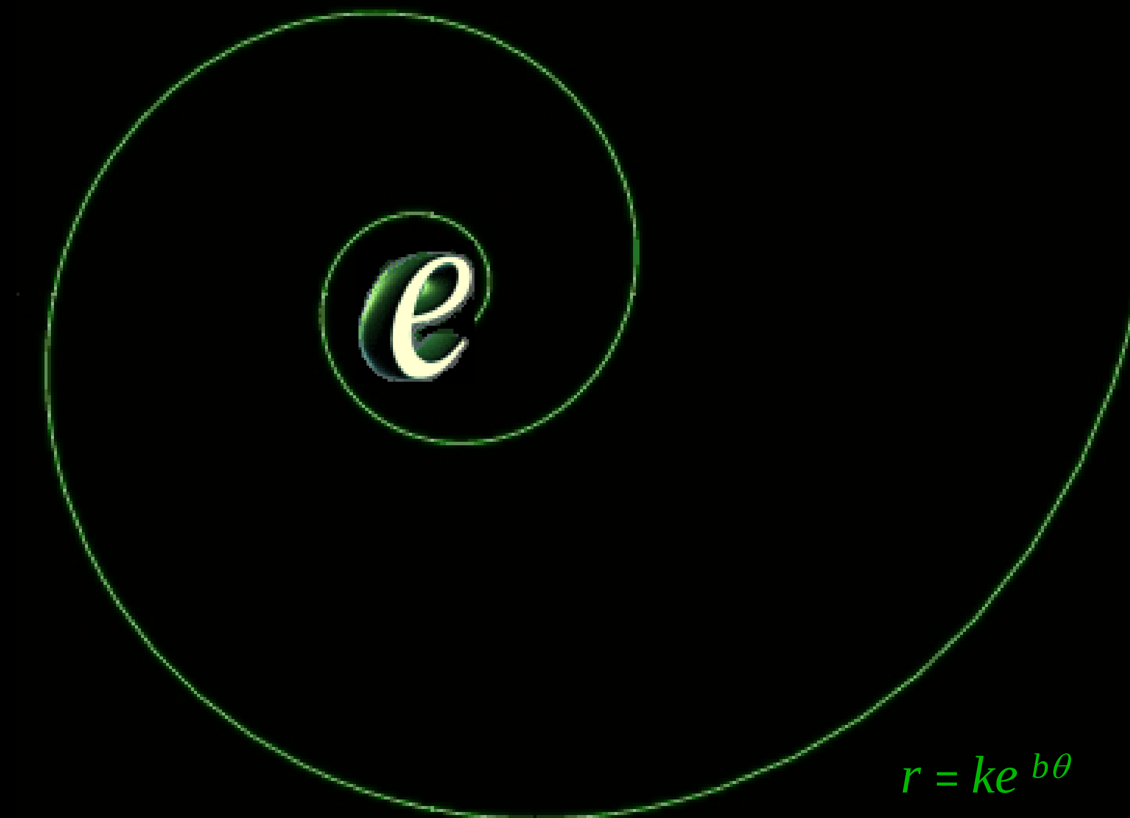
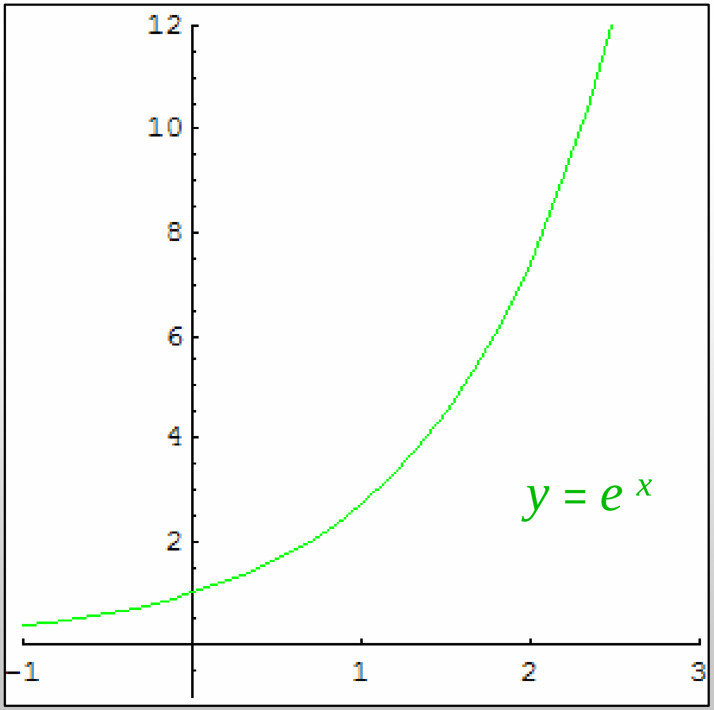


Growth

The number e pops up whenever we examine continuous rates of growth (or decay) that are inherently tied to the amount or size of the thing that we are measuring.

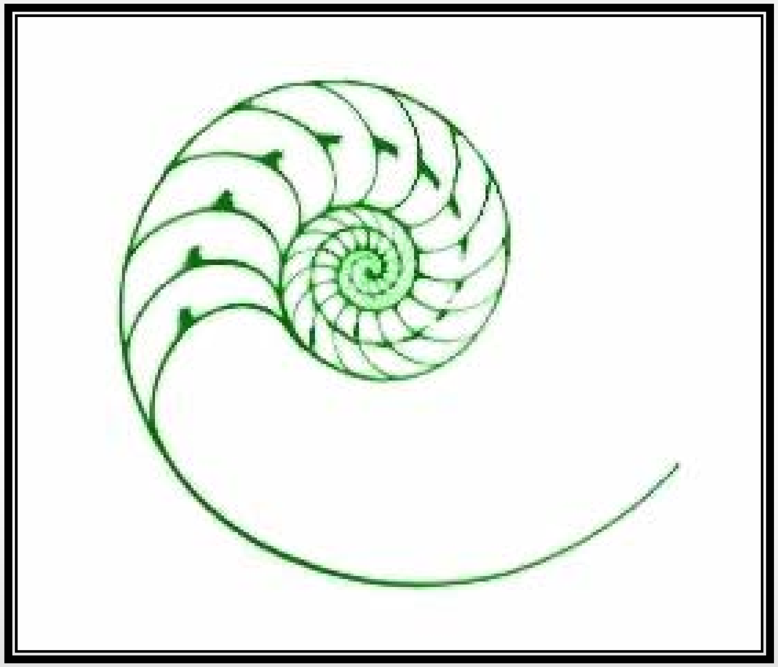
For example, it is used in the calculation of:

- Compound interest
- Population growth
- Radioactive decay
- Bacterial growth
- Atmospheric concentrations of CO_2



Growth

Its presence can also be seen in the cross-section of the chambered nautilus shell which traces out the form of a logarithmic spiral.



Here, the size of each successive chamber is proportional to the one preceding it.

History

Bankers in Europe at the turn of the 17th century knew that the interest they charged grew faster when it was compounded more frequently.

A = total amount

P = principal

r = interest rate per year

Simple interest: $A = P(1 + r)$

Using simple interest, a loan of $1000 at 20% interest per year will require a repayment of

$$\$1000\left(1+\frac{20}{100}\right) = \$1200$$

at the end of one year.

Derivation

If the same loan is compounded twice, we have

$$\$1000\,(1+\tfrac{10}{100}) = \$1100 \quad \text{(end of first six months)}$$

$$\$1100\,(1+\tfrac{10}{100}) = \$1210 \quad \text{(end of second six months)}$$

resulting in \$1210 due at the end of one year. This \$10 more than the amount due with simple interest.

Mathematically, this is equivalent to saying

$$\$1000\,(1+\tfrac{10}{100})\,(1+\tfrac{10}{100}) \quad \text{or,}$$

$$\$1000\,(1+\tfrac{10}{100})^2 = \$1210\,.$$

Derivation

Compounding the same interest four times a year results in a year-end payment of

$$\$1000\,(1+\tfrac{5}{100})^4 = \$1215.51\ .$$

In general, if

t = number of times interest is compounded per year

then

$$A = P\,(1+\frac{r}{t})^t .$$

A natural question is: What is the most money that can be earned at 100% interest? To find out, we set $r = 1$ and see what happens as t increases.

Derivation

t	$(1+\frac{1}{t})^t$ *
1	2
10	2.593742
100	2.704814
1,000	2.716924
10,000	2.718146
100,000	2.718268
1,000,000	2.718280
10,000,000	2.718282

* 6 decimal place accuracy

Value of *e*

What we find is that as t increases, the output of the expression seems to approach a fixed value.

Thus, the payment due at the end of one year on our $1000 loan at 100% interest would be

$$\$1000\ (2.71828) = \$2{,}718.28$$

regardless of whether it is compounded every 32 seconds or every 3.2 seconds.

Value of e

What we have arrived at is the limit definition of e:

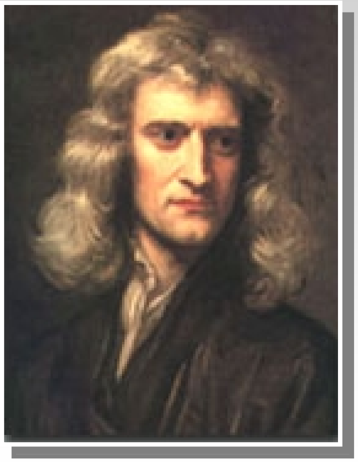
$$e = \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x} \approx 2.718281828459045...$$

Like its better known cousin π, e has special properties:

- It is irrational; it cannot be expressed as a ratio of two integers. The digits to the right of the decimal point continue forever, never falling into a repetitive pattern.
- It is transcendental; using integer coefficients, it is not the solution to any equation of the form

$$ax^{n} + bx^{n-1} + cx^{n-2} ... + d = 0 \; .$$

Direct Method



It was Isaac Newton (1642-1727) who, using the binomial theorem and some clever algebraic manipulation, converted the limit definition of e

$$e = \lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x$$

into an infinite series representation. In 1669, he published what is sometimes referred to as the Direct method:

$$e = \sum_{k=0}^{\infty} \frac{1}{k!} = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \frac{1}{6!} + \dots \ .$$

Summation

The summation sign simply indicates the the quantity to the right of the Σ should be added over the range indicated.

Thus,

$$\sum_{k=1}^{6} k$$

means that we must add together all of the values for k over the range of 1 to 6:

$$\sum_{k=1}^{6} k = 1 + 2 + 3 + 4 + 5 + 6 = \mathbf{21} \;.$$

Factorial

The factorial function is denoted by an exclamation point, “!”, and indicates that a given number *n* should be multiplied by each preceding number from (n-1) down to 1.

n	$n!$
0	**1**
1	**1**
2	2∗1 = **2**
3	3∗2∗1 = **6**
4	4∗3∗2∗1 = **24**
5	5∗4∗3∗2∗1 = **120**
6	6∗5∗4∗3∗2∗1 = **720**
7	7∗6∗5∗4∗3∗2∗1 = **5040**

Factorial

It is important to note two things about the factorial function:

1) These numbers grow very rapidly

2) By definition, for any number n,

$$(n+1)! = (n+1) * n! \quad .$$

For example, with $n = 5$,

$$6! = 6 * 5! = 6 * 5 * 4 * 3 * 2 * 1 \; .$$

Direct Method

Because the denominators of each term increase very rapidly, Newton's series approximation is very efficient at generating the digits of e; the series "converges" quickly.

$$e = \sum_{k=0}^{\infty} \frac{1}{k!} = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \frac{1}{6!} + \ldots$$

Using a large enough value of n, we can calculate the value of e to any desired accuracy:

$$e \approx \sum_{k=0}^{n} \frac{1}{k!} = \frac{1}{1} + \frac{1}{1} + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} + \frac{1}{720} + \ldots + \frac{1}{n!} \;.$$

Direct Method

n	$\sum_{k=0}^{n} \frac{1}{k!}$ *
1	2
2	2.5
3	2.666667
4	2.708333
5	2.716667
6	2.718056
7	2.718254
8	2.718279
9	2.718282

* 6 decimal place accuracy

Accelerating Newton

How can we increase the rate of convergence for this series?

We can try to make the denominators grow even faster by combining pairs of terms. In general, we want to see what happens when we add

$$\frac{1}{n!} + \frac{1}{(n+1)!} \ .$$

Simplifying, we see that these two terms are equivalent to

$$\frac{(n+1)}{(n+1)n!} + \frac{1}{(n+1)!} = \frac{(n+1)}{(n+1)!} + \frac{1}{(n+1)!} = \frac{\boldsymbol{n+2}}{\boldsymbol{(n+1)!}} \ .$$

Accelerating Newton

Using the fundamental characteristic of the factorial function allows us to "compress" consecutive terms into a single term, thereby reducing the number of mathematical operations required to carry out the calculation:

$$\frac{1}{n!} + \frac{1}{(n+1)!} = \frac{n+2}{(n+1)!} \; .$$

Starting with the second term and working backwards gives us an even simpler form:

$$\frac{1}{n!} + \frac{1}{(n-1)!} = \frac{n+1}{n!} \; .$$

Compressed Series

Using our compressed terms and substituting $n = 2k$ (each value of n included two terms) gives us, in the first case,

- $$e = \sum_{k=0}^{\infty} \frac{2k+2}{(2k+1)!} = \frac{2}{1!} + \frac{4}{3!} + \frac{6}{5!} + \frac{8}{7!} + \frac{10}{9!} + \frac{12}{11!} + \frac{14}{13!} + \ldots$$

and in the second case,

- $$e = \sum_{k=0}^{\infty} \frac{2k+1}{(2k)!} = \frac{1}{0!} + \frac{3}{2!} + \frac{5}{4!} + \frac{7}{6!} + \frac{9}{8!} + \frac{11}{10!} + \frac{13}{12!} + \ldots \; .$$

Summing 20 terms, the Direct method yields 18 accurate digits of e. By comparison, these series offer, respectively, 47 and 46 accurate digits.

Compressed Series

It is possible to compress an arbitrary number of terms using the same approach. Here we combine three terms:

$$\frac{1}{n!}+\frac{1}{(n-1)!}+\frac{1}{(n-2)!}=\frac{n^2+1}{n!}$$

resulting in

- $$e=\sum_{k=0}^{\infty}\frac{(3k)^2+1}{(3k)!} = \frac{1}{0!}+\frac{3^2+1}{3!}+\frac{6^2+1}{6!}+\frac{9^2+1}{9!}+\frac{12^2+1}{12!}+\ldots$$

which is accurate to 78 correct digits after 20 terms.

Compressed Series

The alternating series for $1/e$ is derived from the work of the great Swiss mathematician, Leonhard Euler (1707-1783). Substituting -1 into his power series for e^x results in

$$\frac{1}{e} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} = \frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!} - \ldots \ .$$

Compressing it pairwise gives us the decreasing series

- $$\frac{1}{e} = \sum_{k=0}^{\infty} \frac{1-2k}{(2k)!} = \frac{1}{0!} - \frac{1}{2!} - \frac{3}{4!} - \frac{5}{6!} - \frac{7}{8!} - \frac{9}{10!} - \frac{11}{12!} - \ldots$$

which is accurate to 46 digits after 20 terms, over 2½ times the accuracy of the series from which it is derived.

Further Exploration

A whole new family of series expressions for e can be derived by first compressing terms and then manipulating the resulting series in various ways. For instance, adding the first two series we derived gives us a third new series:

$$\left(e = \sum_{k=0}^{\infty} \frac{2k+1}{(2k)!} = \frac{1}{0!} + \frac{3}{2!} + \frac{5}{4!} + \frac{7}{6!} + \ldots \right) \quad +$$

$$\left(e = \sum_{k=0}^{\infty} \frac{2k+2}{(2k+1)!} = \frac{2}{1!} + \frac{4}{3!} + \frac{6}{5!} + \frac{8}{7!} + \ldots \right) \quad =$$

- $$2e = \sum_{k=0}^{\infty} \frac{k+1}{k!} = \frac{1}{0!} + \frac{2}{1!} + \frac{3}{2!} + \frac{4}{3!} + \frac{5}{4!} + \frac{6}{5!} + \frac{7}{6!} + \ldots \, .$$

Further Exploration

Dividing the following compressed series by 2

$$e = \sum_{k=0}^{\infty} \frac{2k+2}{(2k+1)!} = \frac{2}{1!} + \frac{4}{3!} + \frac{6}{5!} + \frac{8}{7!} + \frac{10}{9!} + \frac{12}{11!} + \frac{14}{13!} + \ldots$$

gives us

- $$\frac{e}{2} = \sum_{k=0}^{\infty} \frac{k+1}{(2k+1)!} = \frac{1}{1!} + \frac{2}{3!} + \frac{3}{5!} + \frac{4}{7!} + \frac{5}{9!} + \frac{6}{11!} + \frac{7}{13!} + \ldots \ .$$

Conclusion

In the roughly four centuries since it was discovered, e has revealed itself to be a truly universal constant.

While these new series appear to provide the fastest ways to calculate e, the greatest value of these expressions may lie simply in the process of obtaining them; the methods are exploratory, fun, and within the grasp of anyone with an interest in math.

Finally, these formulas remind us that, even in the case of a subject rigorously studied for over 300 years, students and amateur researchers can make personal discoveries that build directly on the work of giants like Newton.

THe END

Reference Material

Web Resources

- Mathematics research by Harlan Brothers

  http://www.brotherstechnology.com/math/

- Wolfram Research page on *e*

  http://mathworld.wolfram.com/e.html

- NASA Goddard Institute for Space Studies: "Serendipit-*e*"

  http://www.giss.nasa.gov/research/briefs/knox_03/

- Science News - Ivars Peterson: "Hunting *e*"

  http://www.sciencenews.org/articles/20040214/mathtrek.asp/

- UAB Magazine: "To '*e*' or Not To '*e*' ? That's a Constant Question"

  http://main.uab.edu/show.asp?durki=25350

Reference Material

Publications

- H. J. Brothers, "Improving the convergence of Newton's series approximation for *e*." *The College Mathematics Journal*, volume 35, number 1, 2004; pages 34-39.

  http://www.brotherstechnology.com/docs/icnsae_(cmj0104-300dpi).pdf

- J. A. Knox and H. J. Brothers, "Novel series-based approximations to *e*." *The College Mathematics Journal*, volume 30, number 4, 1999; pages 269-275.

  http://www.brotherstechnology.com/docs/cmj_paper1.pdf

- H. J. Brothers and J. A. Knox, "New closed-form approximations to the Logarithmic Constant *e*." *The Mathematical Intelligencer*, volume 20, number 4, 1998; pages 25-29.

  http://www.brotherstechnology.com/docs/mi_paper1.pdf