New Series For π

The following new series were derived using methods to appear in the *College Mathematics Journal* in an article entitled "Improving the Convergence of Newton's Series Approximation for e" by Harlan J. Brothers.

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Here are examples where combining terms improves algebraic convergence but results in an expression that is computationally more intensive than the original.

Original Gregory-Leibnitz series:

$$\pi/4 = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} = \frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \frac{1}{13} - \frac{1}{15} + \ldots \quad \textbf{(1)}$$

Gregory-Leibnitz series combining two terms:

$$\pi/4 = 2\sum_{k=0}^{\infty} \frac{1}{(4k+3)(4k+1)} = \frac{1}{3} + \frac{1}{35} + \frac{1}{99} + \frac{1}{195} + \frac{1}{323} + \frac{1}{483} + \frac{1}{675} + \frac{1}{899} + \ldots \quad \textbf{(2)}$$

Euler's series:

$$\pi^2/6 = \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{1}{1} + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \frac{1}{36} + \frac{1}{49} + \frac{1}{64} + \ldots \quad \textbf{(3)}$$

Euler's series combining two terms:

$$\pi^2/6 = \frac{1}{8}\sum_{k=1}^{\infty} \frac{(4k-1)^2+1}{(2k^2-k)^2} = \frac{5}{4} + \frac{25}{144} + \frac{61}{900} + \frac{113}{3136} + \frac{181}{8100} + \frac{265}{17424} + \ldots \quad \textbf{(4)}$$

$$\pi^2/6 = \frac{1}{4}\sum_{k=1}^{\infty} \frac{1 + 4\sum_{j=0}^{k-1}(5+4(j-1))}{\left(\sum_{j=0}^{k-1}(5+4(j-1))\right)^2} = \frac{5}{4} + \frac{25}{144} + \frac{61}{900} + \frac{113}{3136} + \frac{181}{8100} + \frac{265}{17424} + \ldots \quad \textbf{(5)}$$

Here are examples where combining terms not only improves algebraic convergence but also, to the extent tested, increases computational efficiency.

Series derived from the probability density function (pdf) with mean = 0 and standard deviation = 1:

$$\lim_{a\to\infty} \sum_{n=0}^{\infty} \frac{(-1)^n a^{(2n+1)}}{(2n+1)\, 2^{(n-1)}\, n!} =$$

$$\frac{2a}{(1)(0!)} - \frac{a^3}{(3)(1!)} + \frac{a^5}{(10)(2!)} - \frac{a^7}{(23)(3!)} + \frac{a^9}{(72)(4!)} - \ldots = \sqrt{2\pi} \qquad \textbf{(6)}$$

pdf series combining two terms:

$$\lim_{a\to\infty} \sum_{n=0}^{\infty} \frac{a^{(4n-1)} (a^2 (4n-1) - 4n(4n+1))}{2(4n-1)(4n+1)\, 4^{(n-1)} (2n)!} =$$

$$\frac{2a}{(1)(0!)} + \frac{3a^5 - 20a^3}{(30)(2!)} + \frac{7a^9 - 72a^7}{(504)(4!)} + \frac{11a^{13} - 156a^{11}}{(4576)(6!)} + \frac{15a^{17} - 272a^{15}}{(32640)(8!)} + \ldots = \sqrt{2\pi} \qquad \textbf{(7)}$$

Maximum decimal place accuracy (mdpa) for these two series requires a minimum number of terms, t, where, for $a > 10$, $7a^2/8 < t < 9a^2/10$.

Series based on arcsin(x):

(x=1/2)

$$\pi/6 = \sum_{k=0}^{\infty} \frac{2(2k)!}{4^{2k+1}(2k+1)(k!)^2} =$$

$$\frac{(2)(0!)}{(4)(0!)^2} + \frac{(2)(2!)}{(192)(1!)^2} + \frac{(2)(4!)}{(5120)(2!)^2} + \frac{(2)(6!)}{(114688)(3!)^2} + \frac{(2)(8!)}{(2359296)(4!)^2} + \ldots \qquad \textbf{(8)}$$

Arcsin(x) series combining two terms:

$$\pi/6 = \frac{25}{48} + \sum_{k=1}^{\infty} \frac{((10k+5)(8k+5) - 2k)(4k-1)!!}{2^{(4k+3)}(4k+3)(4k+1)(4k+2)!!} =$$

$$\frac{25}{48} + \frac{(193)(3!!)}{(4480)(6!!)} + \frac{(521)(7!!)}{(202752)(10!!)} + \frac{(1009)(11!!)}{(6389760)(14!!)} + \frac{(1657)(15!!)}{(169345024)(18!!)} + \ldots \qquad \textbf{(9)}$$

("!!" *denotes the double factorial operation:* $n!! = n(n-2)(n-4) \times \ldots$)