Math Bite: Finding *e* in Pascal's Triangle

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Mathematicians have long been familiar with the tidy way in which the nth row of Pascal's triangle sums to 2^n (the top row conventionally labeled as $n = 0$). It is less obvious how the rows behave when we multiply their items.

1	1
1 1	1
1 2 1	2
1 3 3 1	9
1 4 6 4 1	96
1 5 10 10 5 1	2500
1 6 15 20 15 6 1	162000

Let s_n be the product for row n; that is, $s_n = \prod_{k=0}^{n} \binom{n}{k}$. On the right-hand side of the figure above, we see the sequence $\{s_n\}$ grows very quickly. To get a sense of its rate of growth, we can look at the ratios of successive terms, $r_n = s_n/s_{n-1}$. The sequence $\{r_n\}$ itself grows rapidly. Examining the ratios of ratios, a familiar pattern emerges:

n	s_n	$r_n = s_n/s_{n-1} \approx$	$r_n/r_{n-1} \approx$
1	1	1	
2	2	2	2
3	9	4.5	2.25
4	96	10.667	2.370
5	2500	26.042	2.441
6	162000	64.800	2.488
$\vdots$	$\vdots$	$\vdots$	$\vdots$
1000	$\approx 1.68\ 10^{215681}$	$2.49\ 10^{432}$	2.71692

THEOREM. $\lim\limits_{n\to\infty} \dfrac{s_{n+1}/s_n}{s_n/s_{n-1}} = e.$

Proof. By direct calculation we get

$$s_n = (n!)^{n+1} \prod_{k=0}^{n} (k!)^{-2}, \quad n \geq 0$$

$$s_n/s_{n-1} = \frac{n^n}{n!}, \quad n \geq 1, \quad \text{and}$$

$$\frac{s_{n+1}/s_n}{s_n/s_{n-1}} = \left(1 + \frac{1}{n}\right)^n.$$

Given that $\lim\limits_{n\to\infty} \left(1 + \dfrac{1}{n}\right)^n = e$, the result follows. ∎

Summary If s_n is the product of the entries in row n of Pascal's triangle then $(s_{n+1}/s_n)/(s_n/s_{n-1}) = (1 + 1/n)^n$, which has the limiting value e.

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