Mandel-Bach Journey: A Marriage of Musical and Visual Fractals

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Abstract

More than two centuries separate the writing of *The Art of Fugue*, by Johann Sebastian Bach, and the printing of the first graphical representation of the Mandelbrot set. Here we examine Bach's Contrapunctus IX from a fractal perspective and listen as it accompanies a zoom into the Mandelbrot set.

1 Introduction

In the event they might encounter extraterrestrial civilization, the two Voyager spacecraft each carried a 12-inch, gold-plated copper phonograph record containing images and sounds intended to offer a sampling of life and culture on Earth. Dr. Carl Sagan chaired the committee in charge of selecting an appropriately diverse collection of material. It is said that when he asked Dr. Lewis Thomas what should be included, Thomas replied "I think we should send all of Bach; but of course we would be bragging, but it is surely excusable to put the best possible face on at the beginning of such an acquaintance." [12]

The image of Bach's music speeding through interstellar space provides the ending theme for Michael Lawrence's new music documentary *BACH & Friends* [11]. When he first contacted me in November 2008, he had a sense that *The Art of Fugue* was somehow fractal and he wanted to include an appropriate animation. We did not know at the time that the animation would provide a perfect ending for the film.

While I knew I could produce whatever graphics Michael required, a compelling question for me was whether *The Art of Fugue* might indeed possess an inherent power-law relation.

2 Bach and Fractals

Johann Sebastian Bach was a musical master of mathematical manipulation. It is widely appreciated that many of the compositional techniques he used, such as transposition, inversion, and retrograde inversion, have analogs in the world of classical geometry [5, 8]. However, his innate sense of symmetry seems to have been even more profound than the classical analysis of his prodigious output suggests. At least some of his music is known to possess characteristics of fractal geometry [1, 2, 9, 10].

There are several known examples of statistical self-similarity in Bach's music that span his inventions, cello suites, and flute partita [2, 9]. To see how this type of musical self-similarity works, we will examine the distribution of melodic intervals (the difference between consecutive pitches) in his Contrapunctus IX.

From the standpoint of Western music theory, there are 12 different pitches that repeat over a range of just over seven octaves on a grand piano. Unlike pitch, which has an absolute size based on wave period, interval size is perceived relative to its associated pitches. That is to say, a major third (4 semitones) is characterized as such regardless of the octave in which it appears. In Contrapunctus IX, analyzing the collection of intervals modulo 12 reveals a convincing power-law relation.

To establish a power-law, we place the intervals into equal-size bins wherein each bin contains intervals of roughly the same size [4]. In order to maintain consistency with the logarithmic nature of pitch, we assess the size of a melodic interval of k semitones using the function $f(k) = 2^{(k/12)}$. The representative size of a bin is typically assigned using either the lowest, highest, or mean values with the choice applied consistently across all bins.

If N_i represents the number of elements in bin B_i, and s_i represents the assigned size of its elements, then plotting Log(N_i) against Log(s_i) for all B_i reveals the presence of a power-law relation if the plotted points fall on a straight line. Given that there is a heterogenous distribution of elements, we then have evidence of fractal structure. The absolute value of the slope of the regression line for the data is generally taken to be the dimension of the set of elements (for a detailed treatment as applied to music, see Brothers [2]).

Interval	0	1	2	3	4	5	6	7	8	9	10	11
Count	74	549	794	93	69	71	8	32	5	10	7	2

Table 1 : *Raw interval data (mod 12)*

Bin	1	2	3
Count	1510	180	24

Table 2 : *Binned interval data (mod 12)*

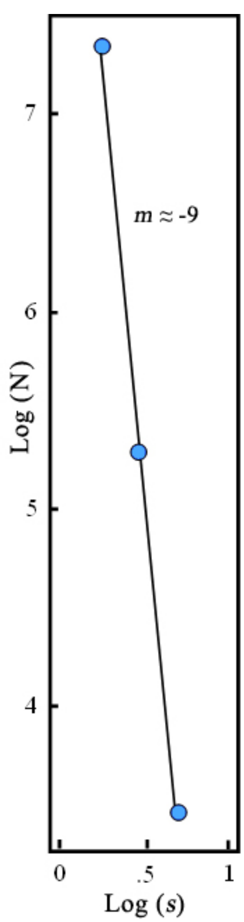

Figure 1 : *Log-log plot of binned data.*

Tables 1 and 2 show the distributions for the raw and binned data respectively. For the purpose of this analysis, each bin spans a range of 4 intervals. The raw data, while somewhat suggestive of a logarithmic relationship, is clearly not monotonic. The binned data, on the other hand, offers a clearer picture. Figure 1 shows the log-log plot of the binned data along with its regression line ($R^2 \approx .9998$). The slope of the line indicates a dimension of $d \approx 9$ with respect to the way in which the pitches change.

It should be noted that similar results can be obtained using a larger number of smaller bins. However, the R^2 values for the respective regression lines are lower and therefore indicate less compelling evidence of a power-law relation as compared to the results using 3 bins.

3 The Graphics

While there is no known mathematical correspondence between the music of Bach and the Mandelbrot set, their structures suggest a poetic connection. Bach's compositions represented the height of Baroque sensibilities; intricacy, nested levels of adornment, and the suggestion of infinite space were compelling structural properties in the art and architecture of his day.

When listening to his music, fans of Bach often refer to a sense of endless spirals or wave-like circles-within-circles. Such descriptions are reminiscent of the infinite, Baroque-like detail of the Mandelbrot set.

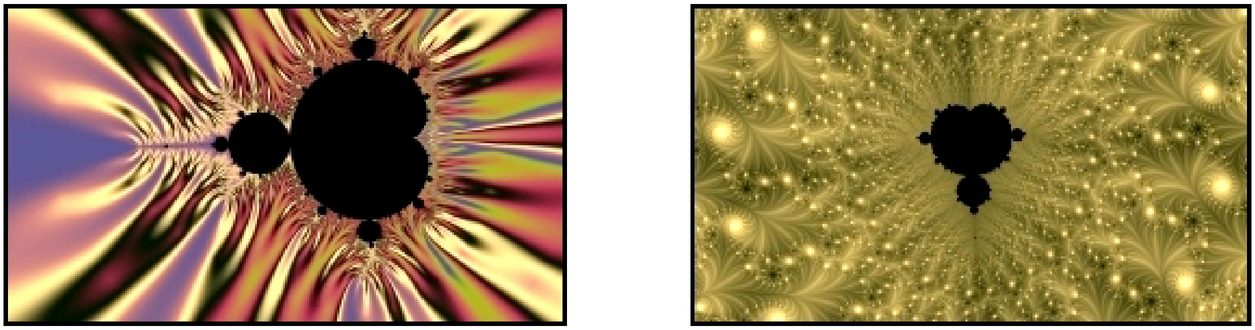

Figure 2 : *On the left is the first frame of the zoom and on the right is the final frame, number 3767.*

The Mandelbrot set thus presented itself as an excellent candidate for echoing something of the sense of *The Art of Fugue*. *UltraFractal* [14] was the software tool used to generate the animation. One of the important design considerations was to be able to eventually enter the heart of a "baby" Mandelbrot set at the end of the zoom.

Beginning with the center coordinate (-0.79598875917681698153, -0.18330667539413717235i) from Jos Leys' film *Dimensions* [13], the big challenge was to choose a suitable coloring algorithm and collection of gradient settings that might suggest a journey through space. After settling on the Triangle Inequality Average for the color function (with a bailout value of 10^{20}), I chose the gradient parameters largely through trial and error. Adjusting these values felt akin to both sculpting and painting; "removing" enough color to offer a feeling of openness, while simultaneously attempting to saturate the remaining colored fields in an engaging manner.

Finally, it was necessary to determine how quickly the magnification factor should change and whether it should accelerate. Progressing too slowly, particularly at the outset, appeared boring. Zooming too quickly left no time for appreciation of the abundant detail and compromised the sense of a smooth transition between frames. Because the amount of detail increases as the animation progresses, a moderate and constant zoom rate seemed appropriate.

4 Adding the Music

Once the animation was rendered, it was time to choose an appropriate fugue for accompaniment. Here, it was important to balance the tempo and dynamics of the music with the perceived speed of the zoom. At the same time, it was necessary that their respective lengths be close enough to ensure that their start and end times could be adjusted to coincide without the need for substantial alteration.

To obtain a high quality, royalty-free rendition of *The Art of Fugue*, I turned to astronomer and musician Jeffrey Hall who had previously arranged and sequenced the work [7]. After listening to the 14 constituent fugues (and four canons), Contrapunctus IX seemed to best embody the correct feel and had roughly the proper duration.[1] It should be noted that this choice was made on an aesthetic basis without regard for any inherent fractal properties.

The tempo (and therefore length) of the audio track was left unaltered. Using a frame rate of 24 fps for the zoom, it required only minor adjustments to set the total number of rendered frames such that the onset of the first and last notes would be synchronized with the appearance of the respective frames.

[1]It is a coincidence that I happened to choose the same piece that Michael Lawrence selected (as performed by the Emerson String Quartet) for the final cut of his documentary.

5 Conclusion

In the resulting animation [3], it is striking to see how the sense of motion we experience presents a perfect counterpoint to the unfolding of Bach's music. Why these seemingly unrelated patterns from the audio and visual domains appear to complement each other so effectively is not immediately clear. Certainly, it was my intent to arrive at a good "fit," and given the fundamental human propensity for finding correlation, even where none exists, it is not surprising that the animation seems to work.

One intriguing possibility is that as humans we are capable of unconsciously discerning musical dimension. There is, indeed, evidence that we are able to register the dimension of graphic objects (see Hagerhall [6]). If this is also true for music, then an awareness of the underlying musical power-law in conjunction with the more explicit visual power-law could contribute to the cohesive quality of the animation.

In the end, perhaps we are simply witnessing the fact that the music of Bach and the graphics of Mandelbrot share the power to intrigue the mind, inspire the spirit, and spark the imagination.

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